REDUCTION OF NIELSEN'S EQUATIONS FOR NONHOLONOMIC MECHANICAL SYSTEMS TO CHAPLYGIN'S EQUATIONS

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Let nonholonomic constraints defined by $k - l$ equations of the form

$$q_\rho^{\cdot} = \sum_{\lambda=1}^{l} a_{\rho\lambda} q_\lambda^{\cdot} + a_\rho \qquad (\rho = l+1, l+2, \ldots, k) \tag{1}$$

be imposed on a mechanical system described by the generalized coordinates $q_1, q_2, \ldots, q_l$ · the generalized forces $Q_1, Q_2, \ldots, Q_k$ and with kinetic energy $T = T(t, q, q^{\cdot})$. Equations of motion of such a system [1] can be written in the form

$$\frac{\partial R_1^*}{\partial q_\lambda^{\cdot}} = N_\lambda \qquad (\lambda = 1, \ldots, l) \tag{2}$$

where the function R_1^* is obtained from

$$R_1 = T^{\cdot} - 2T_0^{\cdot} \tag{3}$$

by replacing all $q_\rho^{\cdot}$ with their expressions given by (1), i.e.

$$R_1^*(t, q_\varkappa q_\lambda^{\cdot}) = R_1(t, q_\varkappa, q_\lambda^{\cdot}, a_{\rho\lambda} q_\lambda^{\cdot} + a_\rho) \begin{pmatrix} \varkappa = 1, \ldots, k \\ \rho = l+1, \ldots, k \end{pmatrix} \tag{4}$$

Symbol T_0 denotes the expression for the kinetic energy T, in which the generalized velccities $q_\varkappa^{\cdot}$ are assumed fixed

$$N_\lambda = Q_\lambda + \sum_{\rho=l+1}^{k} Q_\rho a_{\rho\lambda} \tag{5}$$

We shall show that Eqs. (2) which can be described as reduced Nielsen's equations [2], are reducible to equations given by Voronets [3], while in the case of nonholonomic Chaplygin's systems they reduce to the Chaplygin's equations [4].

Indeed, the relation (4) implies that

$$\frac{\partial R_1^*}{\partial q_\lambda^{\cdot}} = \frac{\partial R_1}{\partial q_\lambda^{\cdot}} + \sum_{\rho=l+1}^{k} \frac{\partial R_1}{\partial q_\rho^{\cdot}} a_{\rho\lambda} \qquad (\lambda = 1, \ldots, l) \tag{6}$$

We shall use the identity

$$\frac{d}{dt}\frac{\partial T}{\partial q_\varkappa^{\cdot}} \equiv \frac{\partial T^{\cdot}}{\partial q_\varkappa^{\cdot}} - \frac{\partial T}{\partial q_\varkappa} \qquad (\varkappa = 1, \ldots, k) \tag{7}$$

and the obvious relations

$$\frac{\partial T_0^{\cdot}}{\partial q_\varkappa^{\cdot}} = \frac{\partial T}{\partial q_\varkappa} \qquad (\varkappa = 1, \ldots, k) \tag{8}$$

which, together with (3), yield (9)

$$\frac{d}{dt}\frac{\partial T}{\partial q_\lambda^{\cdot}} = \frac{\partial R_1}{\partial q_\lambda^{\cdot}} + \frac{\partial T}{\partial q_\lambda}, \qquad \frac{d}{dt}\frac{\partial T}{\partial q_\rho^{\cdot}} = \frac{\partial R_1}{\partial q_\rho^{\cdot}} + \frac{\partial T}{\partial q_\rho} \qquad \begin{pmatrix} \lambda = 1, \ldots, l \\ \rho = l+1, \ldots, k \end{pmatrix}$$

Let us now denote by T^* the expression for the kinetic energy T after the substitution of Eqs. (1), i.e.

$$T^*(t, q_\varkappa, q_\lambda^{\cdot}) = T(t, q_\varkappa, q_\lambda^{\cdot}, a_{\rho\lambda} q_\lambda^{\cdot} + a_\rho) \tag{10}$$

From this we have

$$\frac{\partial T^*}{\partial q_\lambda^{\cdot}} = \frac{\partial T}{\partial q_\lambda^{\cdot}} + \sum_{\rho=l+1}^{k} \frac{\partial T}{\partial q_\rho^{\cdot}} a_{\rho\lambda} \qquad (\lambda = 1, \ldots, l) \tag{11}$$

$$\frac{\partial T^*}{\partial q_\lambda} = \frac{\partial T}{\partial q_\lambda} + \sum_{\rho=l+1}^{k} \frac{\partial T}{\partial q_\rho^{\cdot}} \frac{\partial q_\rho^{\cdot}}{\partial q_\lambda} \qquad (\lambda = 1, \ldots, l) \tag{12}$$

Differentiating (11) with respect to time and subtracting (12), we obtain

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_\lambda^{\cdot}} - \frac{\partial T^*}{\partial q_\lambda} = \frac{d}{dt}\frac{\partial T}{\partial q_\lambda^{\cdot}} - \frac{\partial T}{\partial q_\lambda} + \sum_{\rho=l+1}^{k}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_\rho^{\cdot}}\right) a_{\rho\lambda} + \right.$$
$$\left. + \frac{\partial T}{\partial q_\rho^{\cdot}}\left(a_{\rho\lambda}^{\cdot} - \frac{\partial q_\rho^{\cdot}}{\partial q_\lambda}\right)\right] \tag{13}$$

which, together with (9) and (6), yields an expression allowing us to write the reduced Nielsen's equations (2) in the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_\lambda^{\cdot}} - \frac{\partial T^*}{\partial q_\lambda} + \sum_{\rho=l+1}^{k} \frac{\partial T}{\partial q_\rho^{\cdot}}\left(\frac{\partial q_\rho^{\cdot}}{\partial q_\lambda} - a_{\rho\lambda}^{\cdot}\right) - \sum_{\rho=l+1}^{k} \frac{\partial T}{\partial q_\rho} a_{\rho\lambda} = N_\lambda \tag{14}$$

in which they coincide with those obtained by Voronets. It is clear, that, in the case of Chaplygin's systems, i.e. when $a_\rho \equiv 0$, the coefficients $a_{\rho\lambda}$, the generalized forces and the kinetic energy are independent of the generalized coordinates $q_{l+1}, q_{l+2}, \ldots, q_k$, Eqs. (14) become the Chaplygin's equations

$$\frac{d}{dt}\frac{\partial T^*}{\partial q_\lambda^{\cdot}} - \frac{\partial T^*}{\partial q_\lambda} + \sum_{\rho=l+1}^{k} \frac{\partial T}{\partial q_\rho^{\cdot}}\left[\sum_{\mu=1}^{l}\left(\frac{\partial a_{\rho\mu}}{\partial q_\lambda} - \frac{\partial a_{\rho\lambda}}{\partial q_\mu}\right) q_\mu^{\cdot}\right] = Q_\lambda$$
$$(\lambda = 1, 2, \ldots, l)$$

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